Objectives:

- Find derivatives of composite functions using the Chain Rule.
- Use the Chain Rule to prove the Quotient Rule.

The Chain Rule:
If f and g are differentiable functions:

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

Examples: Find the derivatives of the following functions.

1. $h(x) = \left(x^3 + 2x\right)^{30}$

   Inside function: $z = g(x) = x^3 + 2x$

   Outside function: $f(z) = z^{30}$

   $g'(x) = 3x^2 + 2$ and $f'(z) = 30z^{29}$

   The derivative we want: $h'(x) = f'(g(x))g'(x) = 30(g(x))^{29}g'(x) = 30\left(x^3 + 2x\right)^{29}\left(3x^2 + 2\right)$

2. $h(x) = e^{4-3x}$

   Inside function: $z = g(x) = 4 - 3x$

   Outside function: $f(z) = e^z$

   $g'(x) = -3$ and $f'(z) = e^z$

   The derivative we want: $h'(x) = f'(g(x))g'(x) = e^{g(x)}g'(x) = e^{4-3x}(-3)$

3. $h(x) = \left(x - 3x^2\right)^5$

$$h'(x) = 5\left(x - 3x^2\right)\left(1 - 6x\right)$$

4. $f(x) = \sqrt{4x^2 - 7x + 1} = \left(4x^2 - 7x + 1\right)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}\left(4x^2 - 7x + 1\right)^{\frac{-1}{2}}(8x - 7) = \left(\frac{1}{2}\right)\frac{8x - 7}{\sqrt{4x^2 - 7x + 1}}$$

5. $y = 6\sin(5x^2)$
   Inside: $5x^2$ with derivative $10x$
   Outside: $6\sin(z)$ with derivative $6\cos(z)$

$$y' = 6\cos(5x^2)(10x)$$

6. $s(t) = \sqrt[3]{t^2\tan(t)} = (t^2\tan(t))^{\frac{1}{3}}$

$$s'(t) = \frac{1}{3}(t^2\tan(t))^{-\frac{2}{3}}\frac{d}{dt}\left(t^2\tan(t)\right) = \frac{1}{3}(t^2\tan(t))^{-\frac{2}{3}}\left(t^2\sec^2(t) + 2t\tan(t)\right)$$
$$= \frac{1}{3}(t^{-\frac{4}{3}}\tan^{-\frac{2}{3}}(t))\left(t^2\sec^2(t) + 2t\tan(t)\right) = \frac{1}{3}t^{\frac{2}{3}}\tan^{-\frac{2}{3}}(t)\sec^2(t) + \frac{2}{3}t^{-\frac{1}{3}}\tan^2(t)$$

7. $f(x) = \sec\left(\sqrt{x}\right) = \sec\left(x^{\frac{1}{2}}\right)$

   Inside: $g(x) = x^{\frac{1}{2}}$

   Outside: $h(z) = \sec(z)$

   $g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ and $h'(z) = \sec(z)\tan(z)$

   $f'(x) = h'(g(x))g'(x) = \sec(g(x))\tan(g(x))g'(x) = \sec\left(x^{\frac{1}{2}}\right)\tan\left(x^{\frac{1}{2}}\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{\sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right)}{2\sqrt{x}}$

8. $f(x) = \left(\frac{3-2x}{x^2+1}\right)^4$

$$f'(x) = 4\left(\frac{3-2x}{x^2+1}\right)^3\frac{d}{dx}\left(\frac{3-2x}{x^2+1}\right) = 4\left(\frac{3-2x}{x^2+1}\right)^3\left(\frac{(x^2+1)(-2)-(3-2x)(2x)}{(x^2+1)^2}\right)$$

9. $f(z) = 2^{1/z}$

$$f'(z) = \ln(2)2^{1/z}\frac{d}{dz}\left(\frac{1}{z}\right) = \ln(2)2^{1/z}(-z^{-2}) = \frac{-\ln(2)2^{1/z}}{z^2}$$

Proving the Quotient Rule

When we introduced the Product and Quotient Rules, we proved the Product Rule but not the Quotient Rule. Using the Chain Rule, we can prove that the Quotient rule is true as well.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\left(g(x)\right)^{-1}\right) = g(x)^{-1}\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)^{-1}$$
$$= g(x)^{-1}f'(x) + f(x)\left(-1(g(x))^{-2}g'(x)\right) = \frac{f'(x)}{g(x)} + \frac{-f(x)g'(x)}{(g(x))^2}$$
$$= \frac{g(x)f'(x)}{(g(x))^2} + \frac{-f(x)g'(x)}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Examples Using the Chain Rule Repeatedly:

1. $f(x) = \sin\left(\cos(x^2)\right)$. Find $\frac{d}{dx}f(x)$.

$$f'(x) = \cos\left(\cos(x^2)\right)\frac{d}{dx}\cos(x^2) = \cos\left(\cos(x^2)\right)\left(-\sin(x^2)(2x)\right) = -2x\cos\left(\cos(x^2)\right)\sin(x^2)$$

2. $g(z) = \left(2^{z^5+2}\right)^3$

$$g'(z) = 3\left(2^{x^5+2}\right)^2\frac{d}{dx}\left(2^{x^5+2}\right) = 3\left(2^{x^5+2}\right)^2\left(\ln(2)2^{x^5+2}\right)\frac{d}{dx}(x^5+2)$$
$$= 3\left(2^{x^5+2}\right)^2\left(\ln(2)2^{x^5+2}\right)(5x^4) = 15\ln(2)\left(2^{x^5+2}\right)^3 x^4$$